

TEMPLATES AND TRAIN TRACKS

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ABSTRACT. Within the context of Smale flows on compact manifolds, this article deals with a relationship between abstract templates, branched 1-manifolds (train tracks), and laminations representing unstable separatrices of basic sets. We show that an abstract template, the richest in information of the above three entities, determines a member of each of the remaining two groups, and partial determinations in other directions are developed. As a result of this relationship, an obstruction to the realization of certain abstract templates in nonsingular Smale flows on homology 3-spheres is raised.

1. Introduction. Smale flows on 3-manifolds have come under study recently, with results by Franks and Williams among many others. Franks' *Nonsingular Smale flows on S^3* [6] answered many questions and, unsurprisingly, prompted more. The major question answered was: What combinations of 1-dimensional basic sets can occur in Smale flows on S^3 ? The pertinent information is contained in a Lyapunov graph—a graph where vertices are labeled with chain-recurrent pieces of a flow on a compact space. In fact, the label of a suspension of a subshift of finite type in the above article is a nonnegative matrix. In the present paper, we begin an investigation into another symbol for these basic sets: A signed matrix (structure matrix) with certain order data included. Eventually, we shall try to determine which collections of these can be realized in S^3 ; so far, we just notice that in the above context, an order array A (definition within) determines a unique abstract ordered branched 1-manifold K : a smooth graph where the order of branches at each branch point matters. K then determines an orientable surface of unique minimal genus g ; that is, to embed K in a surface S , the genus of S must be $\geq g$. When a basic set Λ corresponding to A is embedded in a nonsingular flow on a compact oriented 3-manifold M with $H_1(M; \mathbb{Q}) = 0$, K must be embedded in a surface N which is transverse to the flow on M ; because such N must be tori, we have

THEOREM A. *A necessary condition for A to be realized in a nonsingular flow on a homology 3-sphere is that $g \leq 1$.*

An example of a basic set whose genus (in the sense of the above paragraph) is greater than 1 appears after Theorem 4.3.

As we approach the above issue, we notice that the branched 1-manifolds are fundamental to the following problems: (1) For 1-dimensional basic sets Λ in flows on 3-manifolds, describe $W^u(\Lambda) \cap \partial X$, where X is an isolating neighborhood of Λ , and (2) To what extent does an abstract ordered branched 1-manifold determine a

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1-dimensional basic set? We have

THEOREM B. *Assume Λ is a basic set of a Smale flow on a compact orientable 3-manifold M corresponding to an order array A with derived branched 1-manifold K , and g is a Lyapunov function (defined within) for M such that $g(\Lambda) = c$. If S is a component of $g^{-1}(c - \varepsilon)$ for ε sufficiently small, $W^u(\Lambda) \cap S$ is a finite number of circles together with sets of the form $C \times \mathbf{R}$, C a Cantor set, where $C \times t$ approaches a circle asymptotically as $t \rightarrow \infty$ and likewise for $t \rightarrow -\infty$; furthermore, K carries $W^u(\Lambda) \cap S$. An analogous result holds for $W^s(\Lambda)$.*

We use a combing procedure discussed by Williams in [8] to introduce an equivalence relation on a set of branched 1-manifolds and produce

THEOREM C. *An equivalence class of branched 1-manifolds with pairing and squeezing data (defined herein) determines a unique flow equivalence class of the associated order arrays, possibly the empty class.*

The above results shed a bit more light on Smale flows in dimension 3; in particular, they aid in providing a naive schematic picture of the situation. By using templates to approximate 1-dimensional basic sets, the results of Theorem B are evident, and we imagine constructing Smale flows by a series of identifications, gluing template boundaries—branched 1-manifolds—to surfaces, using rules delineated in a Lyapunov graph.

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2. Background and definitions. We will assume throughout, unless explicitly stating otherwise, that ϕ_t is a smooth flow on a compact manifold M .

DEFINITION 2.1. If ϕ_t is a flow on M and $x \in M$, we say that x is *chain-recurrent* if for any $\varepsilon > 0$ there exist points $x_1 = x, x_2, \dots, x_n = x$ and real numbers $t(i) \geq 1$ such that $d(\phi_{t(i)}(x_i), x_{i+1}) < \varepsilon$ for all $1 \leq i < n$. The set of all such points, called the chain-recurrent set R , is a compact invariant set under the flow.

DEFINITION 2.2. If ϕ_t is a flow on M , then a smooth function $g: M \rightarrow \mathbf{R}$ is called a *Lyapunov function* provided (a) $d(g(\phi_t(x)))/dt < 0$ for $x \notin R$, the chain-recurrent set; (b) For $x, y \in R$, $g(x) = g(y)$ if and only if for each $\varepsilon > 0$ there exist points $x_1 = x, x_2, \dots, x_n = y, x_{n+1}, \dots, x_{2n} = x$ in R and real numbers $t(i) > 1$, $1 \leq i < 2n$, such that $d(\phi_{t(i)}(x_i), x_{i+1}) < \varepsilon$, $1 \leq i < 2n$.

Results of [2 and 10] imply that any smooth flow on a compact manifold possesses a Lyapunov function.

A compact invariant set Λ of a flow ϕ_t on M has a *hyperbolic structure* if there is a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u \oplus E^c$ preserved by $D\phi_t$ for all t , where E^c is spanned by the vector field tangent to the flow, and there exist constants $C, a > 0$ such that

$$\begin{aligned} \|D\phi_t(v)\| &\leq Ce^{-at}\|v\| \quad \text{for } v \in E^s, t \geq 0, \\ \|D\phi_t(v)\| &\geq C^{-1}e^{at}\|v\| \quad \text{for } v \in E^u, t \geq 0. \end{aligned}$$

THEOREM 2.3 [7]. *If the chain-recurrent set R of a flow ϕ_t on a compact manifold M has a hyperbolic structure, R is a finite disjoint union of compact*

invariant sets $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ and each Λ_i has a point whose forward orbit is dense in Λ_i .

The Λ_i are called *basic sets* for the flow; by Definition 2.2(b), any Lyapunov function is constant on each Λ_i .

PROPOSITION 2.4 [6]. *Suppose that ϕ_t is a flow on a compact oriented manifold M with Lyapunov function $g: M \rightarrow \mathbf{R}$ and that R has finitely many chain-transitive pieces. If Γ is the quotient space of M obtained by identifying each component of a level set of g to a point, then Γ is a finite graph; furthermore, if $H_1(M; \mathbf{Q}) = 0$, Γ is a tree.*

Γ is called a *Lyapunov graph* if we label each vertex with its chain-transitive piece.

DEFINITION 2.5. For $x \in \Lambda$, a compact invariant hyperbolic set for ϕ_t , define the *stable* and *unstable manifold* of x :

$$W^s(x) = \{y \in M \mid \text{for some } r \in \mathbf{R}, d(\phi_t(y), \phi_{t+r}(x)) \rightarrow 0 \text{ as } t \rightarrow \infty\},$$

$$W^u(x) = \{y \in M \mid \text{for some } r \in \mathbf{R}, d(\phi_t(y), \phi_{t+r}(x)) \rightarrow 0 \text{ as } t \rightarrow -\infty\};$$

we say $W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x)$ and $W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x)$.

We say that a flow ϕ_t on M with hyperbolic chain-recurrent set R satisfies the *strong transversality condition* if for each $x, y \in R$, $W^s(x) \pitchfork W^u(y)$ (transverse intersection).

DEFINITION 2.6. A smooth flow ϕ_t on a compact manifold M is called a *Smale flow* provided:

- (1) its chain-recurrent set R has a hyperbolic structure and $\dim R \leq 1$;
- (2) it satisfies the strong transversality condition.

For Smale flows, Bowen has completely categorized the dynamics on each basic set, up to topological equivalence.

THEOREM 2.7 [1]. *Suppose ϕ_t is a flow with hyperbolic chain-recurrent set and Λ is a 1-dimensional basic set. Then $\phi_t|_\Lambda$ is topologically equivalent to the suspension of a subshift of finite type with an irreducible matrix.*

We are then prompted to discuss suspensions of subshifts of finite type. Given an $n \times n$ nonnegative integer matrix A of zeroes and ones, let

$$\Sigma_A = \left\{ \underline{x} \in \prod_{-\infty}^{\infty} \{1, \dots, n\} \mid A_{x_k x_{k+1}} = 1 \text{ for all } k \right\}$$

and define the right shift map $\sigma_A: \Sigma_A \rightarrow \Sigma_A$ by $\sigma_A(\underline{x}) = \underline{y}$ where $y_k = x_{k-1}$. Then Σ_A is a compact 0-dimensional space and σ_A is called a *subshift of finite type*; it has *topological transitivity* if it has a dense orbit, and this is so if A is *irreducible*: for each i, j with $1 \leq i \leq n$, $1 \leq j \leq n$ there is an $N > 0$ such that $(A^N)_{ij} \neq 0$.

DEFINITION 2.8. If $h: X \rightarrow X$ is a homeomorphism then its *suspension flow* is defined on the identification space $Y = X \times \mathbf{R}/(x, s+1) \sim (h(x), s)$, and is defined to be the flow on Y induced by the flow ϕ_t on $X \times \mathbf{R}$ given by $\phi_t(x, s) = (x, s+t)$. If $X = \Sigma_A$, we write $Y = X_A$.

Different matrices can represent equivalent dynamics:

DEFINITION 2.9. Two nonnegative integer matrices A and B are said to be *flow equivalent* provided the suspended subshifts of finite type X_A and X_B are topologically equivalent.

A useful result is

THEOREM 2.10 [5]. Suppose that A and B are nonnegative irreducible integer matrices neither of which is a permutation matrix. Then A and B are flow equivalent if and only if: (1) $\det(I_n - A) = \det(I_m - B)$, and (2) $Z^n/(I_n - A)Z^n \cong Z^m/(I_m - B)Z^m$, where n and m are the sizes of A and B respectively and I_n and I_m are identity matrices.

Suppose that $U \subset V$ are compact surfaces (with boundary) and that $f: U \rightarrow V$ is a diffeomorphism onto its image. Let $H = \bigcup_i h_i$ be a finite set of 1-handles in U —that is, each h_i is an embedded copy of $D^1 \times D^1$ where $D^1 = [-1, 1]$.

If x is a point of $D^1 \times p \subset h_i$, $p \in D^1$, we write $W_i^u(x) = D^1 \times p$; similarly $W_i^s(x)$ will denote the interval $q \times D^1$ in h_i containing x .

DEFINITION 2.11. The embedding $f: U \rightarrow V$ is *hyperbolic with respect to the set of handles H* provided:

(1) if $x \in h_i$ and $f(x) \in h_j$, then $\text{int}(f(W_i^u(x))) \supset W_j^u(f(x))$ and $f(W_i^s(x)) \subset \text{int}(W_j^s(f(x)))$; and

(2) there is a $\lambda \in (0, 1)$ such that for each $x \in h_i$ with $f(x) \in H$ and each $v \in T_x(W_i^s(x))$, $w \in T_x(W_i^u(x))$ we have $\|Df(v)\| \leq \lambda\|v\|$ and $\|Df(w)\| \geq \lambda^{-1}\|w\|$.

We then see that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(H)$ has a hyperbolic structure and that $W_i^s(x)$ and $W_i^u(x)$ are contained in the stable and unstable manifolds of x respectively.

DEFINITION 2.12. The *geometric intersection matrix* G associated to f and H is defined by G_{ij} = number of components of $f(h_j) \cap h_i$ = number of points of intersection of $W_i^s(x)$ and $f(W_j^u(y))$ for any $x \in h_i$, $y \in h_j$.

Then the suspension of $f|_\Lambda$ is topologically equivalent to the suspension of the subshift of finite type $\sigma_G: \Sigma_G \rightarrow \Sigma_G$ (see [6]).

If Λ_i is a basic set corresponding to a hyperbolic handle set H , we call $K(\Lambda_i) = \bigcup_{t \in \mathbb{R}} \phi_t(H)$ the *saturated handles*.

We define a *strip* k to be an embedded 1-handle; images of $D^1 \times p$ are in the *core foliation* of k , and images of $q \times D^1$ in its *transverse foliation*. We call the images of $\{-1\} \times D^1$ and $\{1\} \times D^1$ the *ends* of k .

DEFINITION 2.13. A foliated *train track neighborhood* N in a surface V is a finite union of strips $\bigcup k_i$ such that

(1) if $k_i \cap k_j \neq \emptyset$, it consists of an end of k_i either embedded in the interior of, or equal to, an end of k_j . Each end of a strip intersects some other strip;

(2) the foliations on the interiors of all the handles taken together extend to foliations on $\text{int } N$.

PROPOSITION 2.14 [6]. Let ϕ_t be a Smale flow on a 3-manifold M with Lyapunov function g , and suppose Λ is a basic set of ϕ_t associated to a handle set H with $g(\Lambda) = c$. If $S =$ a component of $g^{-1}(c - \varepsilon)$ for ε sufficiently small, and S and Λ are in the same component of $g^{-1}([c - \varepsilon, c + \varepsilon])$, then $N^- = S \cap K(\Lambda)$ is a foliated train track neighborhood. An analogous result holds for P , an appropriate component of $g^{-1}(c + \varepsilon)$.

Williams' very useful idea of employing branched manifolds in this field plays an integral role here—we consider branched manifolds of dimensions 1 and 2: a *branched 1-manifold* or *train track* is a 1-dimensional subcomplex K of a surface F with the following properties.

- (1) The branches (edges) are C^1 and are tangent at the vertices.
- (2) At least 3 edges emanate from each vertex.
- (3) Each vertex lies in the interior of a C^1 arc embedded in K .

By an *abstract (ordered) branched 1-manifold*, we will mean a 1-dimensional subcomplex K of S^3 where a neighborhood (in K) of each vertex is smoothly embedded in S^2 , along with properties 1–3 above. A fixed orientation of S^2 gives a cyclic order on the edges of K at each branch point (vertex).

We shall consider only those branched 1-manifolds with a (locally) single-sheeted side: On at least one (of the two) side of each branch point x there is only one branch connected to x . We shall occasionally refer to this single-sheeted side as a 0-branch.

A lamination L in a surface F (a foliation of a closed subset of F) is *carried* by a train track K if L and K are C^1 -close.

We shall now discuss branched 2-manifolds, herein restricted to the concept of abstract templates. Consider a block T_i as in Figure 2.1. We have drawn a partial flow on T_i which is given by the unit downward vector field. The flow exits T_i on the β_i 's and the “arch intervals” between them, and is tangent to the vertical sides.

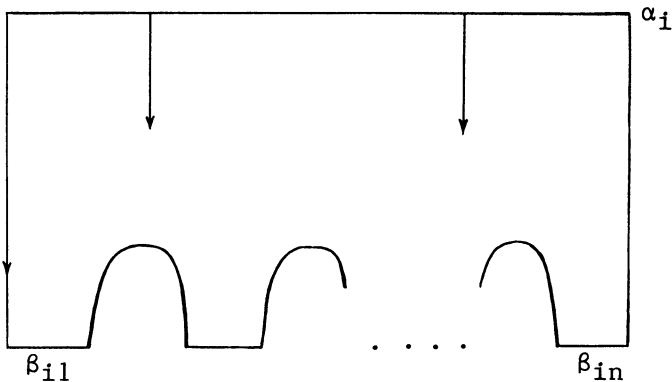


FIGURE 2.1

We form a branched surface T , with boundary, by gluing together a finite number of blocks T_i , attaching each β_{ij} to an α_k and vice versa. We do this in such a way that each attaching map is overflowing and expanding; that is, we require α_k to be in the interior of β_{ij} and that the attaching map from any interior subinterval of β_{ij} to α_k is strictly expanding. We also contrive to fit together the vector fields on each T_i to yield a (partial) semiflow ϕ_t on $T = \bigcup_i T_i$. See Figure 2.2.

DEFINITION 2.15. A branched surface T with semiflow ϕ_t constructed as above is called an *abstract template*. The α_i are called *branch lines*.

Consider an $f: U \rightarrow V$ which is hyperbolic with respect to a handle set $H = \bigcup_i h_i$. We modify H and f so that for any h_i , $f(h_i) \cap h_j$ is connected $\forall j$; using this method Figure 2.3a becomes Figure 2.3b. This creates more 1-handles in H

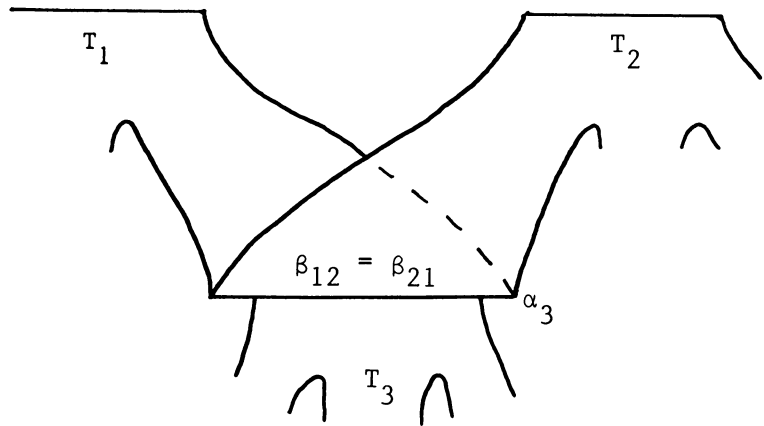


FIGURE 2.2

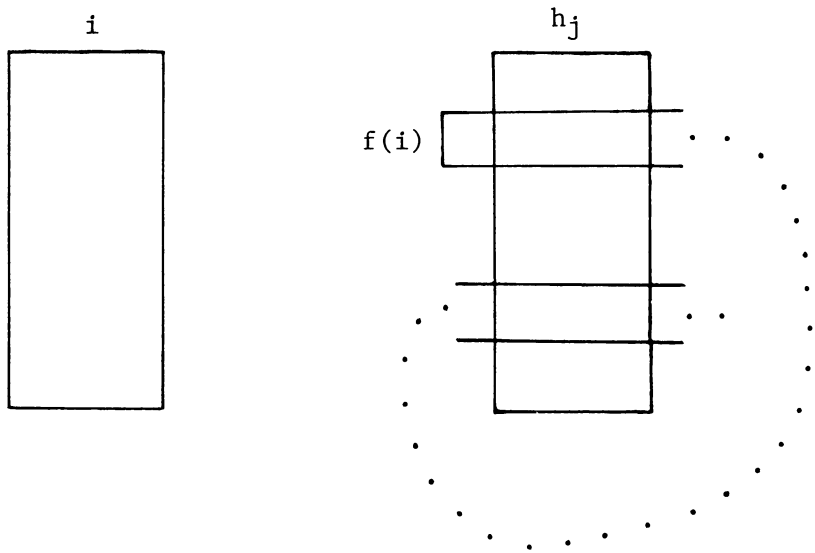


FIGURE 2.3a

but leaves the dynamics unchanged, for we have discarded only wandering points. Suppose we now have n handles.

Next we assign some orientation data to each 1-handle: left-to-right for the horizontal direction and top-to-bottom for the vertical direction. We consider the horizontal direction as unstable, and the vertical as stable; choosing the unstable direction as primary, this yields a 2-dimensional orientation for each h_i .

Then given h_i , $f(h_i)$ hits k_i handles, and we count these intersections from $f(\text{left end of } h_i)$ to $f(\text{right end of } h_i)$ using the assigned orientations. We say that the numbers $1, \dots, k_i$ are *horizontal crossing numbers* of h_i . Similarly, given h_j , m_j handles hit h_j under f , and we count these intersections from the top of h_j to the bottom, again using assigned orientations. We call the numbers $1, \dots, m_j$ *vertical crossing numbers* of h_j .

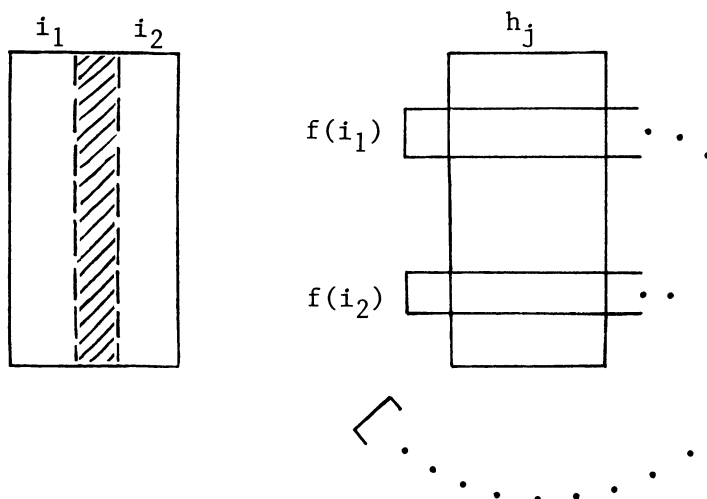


FIGURE 2.3b

We now create an $n \times n$ array (a_{ij}) whose entries are either 0 or ordered triples: $a_{ij} = 0$ if $f(h_j) \cap h_i = \emptyset$; the first entry in a_{ij} is 1 (respectively -1) if $f(h_j) \cap h_i \neq \emptyset$, with the unstable direction preserved (respectively reversed) by f ; if $a_{ij} \neq 0$, its second entry is the horizontal crossing number of h_j for $f(h_j) \cap h_i$; if $a_{ij} \neq 0$, its third entry is the vertical crossing number of h_i for $f(h_j) \cap h_i$.

DEFINITION 2.16. An array A constructed as above is called an *order array* for H and f .

Lastly we introduce a matrix representation of a branched 1-manifold. We assume that an orientation has been assigned to each branch point—this would certainly be the case if the train track were embedded in an orientable surface. We then have locally, Figure 2.4.

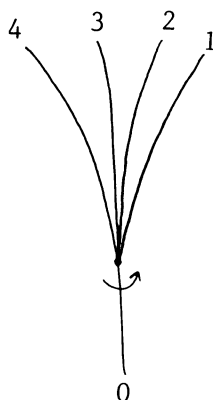


FIGURE 2.4

Now we assume that the train track has n branch points. We will create an $n \times n$ matrix B with each entry b_{ij} consisting of a finite collection of symbols representing paths between branch points: b_{ij} will contain $\binom{h}{m}$ if there is a path from branch pt. $\#i$ (branch h) to branch pt. $\#j$ (branch m).

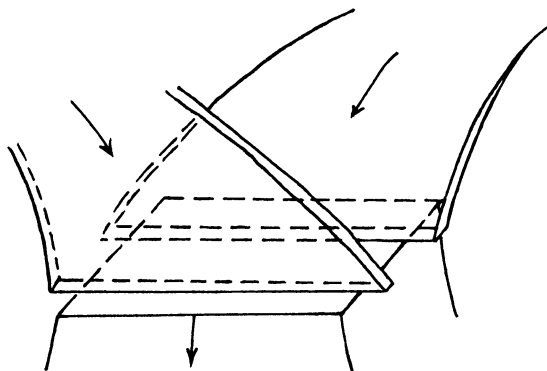


FIGURE 3.1

Note that B is skew-symmetric, in a sense:

If

$$b_{ij} = \begin{pmatrix} h \\ m \end{pmatrix}, \begin{pmatrix} l \\ p \end{pmatrix}, \dots, \begin{pmatrix} k \\ q \end{pmatrix}$$

then

$$b_{ji} = \begin{pmatrix} m \\ h \end{pmatrix}, \begin{pmatrix} p \\ l \end{pmatrix}, \dots, \begin{pmatrix} q \\ k \end{pmatrix}.$$

For diagonal entries (paths from a branch point to itself), we choose the greater of the two branch numbers for the “superscript.”

DEFINITION 2.17. The matrix B in the above discussion is called a *branched 1-manifold matrix*.

3.

A. *Branched 1-manifolds of handle sets.* We now begin to describe the relationship involving templates, branched 1-manifolds, and laminations mentioned in the Introduction. Firstly we discuss a connection between abstract templates and handle sets.

LEMMA 3.1. *There is a 1-1 correspondence between the set of handle sets and the set of abstract templates.*

PROOF. Given an abstract template T we form its associated thickened template T' by thickening T in a transverse direction (in S^3). We do this in such a way that the (partial) semiflow on T becomes a (partial) flow with hyperbolic structure of index 1 on T' : Branch lines become branch rectangles, and incoming sheets meeting at a branch line become disjoint incoming thickened sheets joined to a branch rectangle. See Figure 3.1. The flow on T' contracts in the new (formerly transverse) direction.

The associated handle set consists of the branch rectangles; i.e., the handle set is a cross section to the partial flow on T' , and f is the first-return map.

Conversely, suppose we have a handle set H . Then there is an $f: H \rightarrow f(H) \subset V$, a compact neighborhood of H . We extend this to $\bar{f}: V \rightarrow V$ and form its suspension flow ϕ_t . Restricting the domain of ϕ_t to H and flowing forward gives us a thickened template, and collapsing this in the stable (contracting) direction yields a template. Q.E.D.

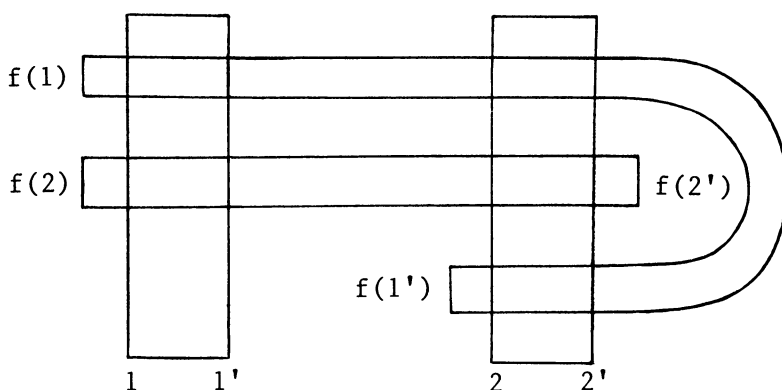


FIGURE 3.2

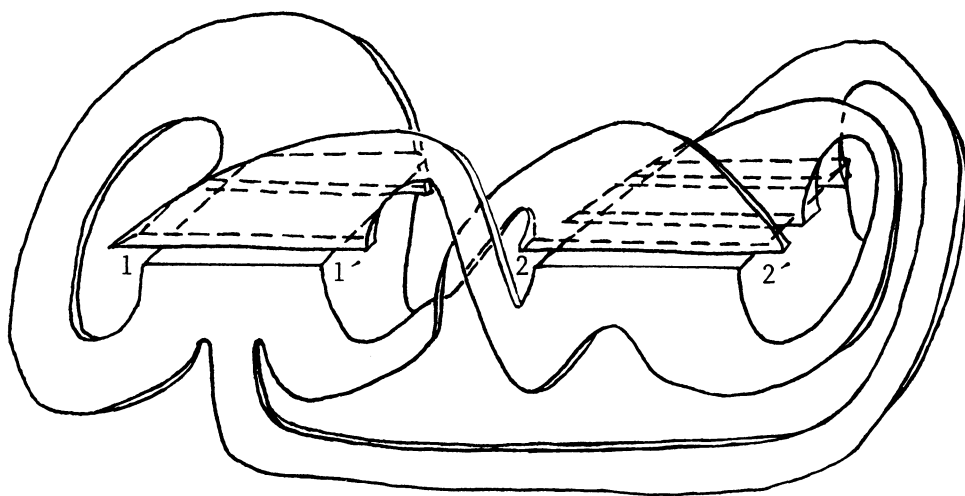


FIGURE 3.3

For example, consider the handle set in Figure 3.2. This becomes the thickened template of Figure 3.3.

There are now, because of Lemma 3.1, two equivalent ways to obtain an ordered branched 1-manifold from a handle set:

- (1) as the boundary of the associated template, and
- (2) from the train track neighborhood given in Proposition 2.14: collapsing each transverse leaf of this neighborhood to a point yields a branched 1-manifold.

PROPOSITION 3.2. *Let A be an order array representing a basic set Λ in a Smale flow on an oriented 3-manifold M with Lyapunov graph Γ . Then A determines two ordered branched 1-manifolds, one labeled “backward,” the other “forward.”*

PROOF. A is the label of a vertex x in Γ ; we restrict our attention to a connected neighborhood of x which contains no other vertex. The branched 1-manifold L_f derived from A is then embedded in an oriented (because M is) surface S which is represented by a point(s) on an edge(s) just “below” x in Γ . Thus, L_f is an ordered branched 1-manifold, and is called the “forward” ordered branched 1-manifold. By

considering the reverse flow, A yields an “inverse” order array A' , and the above reasoning gives L_b , the “backward” ordered branched 1-manifold. Q.E.D.

Notes. (1) The surfaces and branched 1-manifolds mentioned above need not be connected.

(2) The author has developed an algorithm for deriving a branched 1-manifold matrix for L_f directly from A ; likewise for L_b and A' . It is cumbersome, but a computer program could be written for it. See Theorem 3.2 of [3].

(3) In a branched 1-manifold matrix obtained from an order array, there are no occurrences of $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This follows from the stipulation in Definition 2.15 that each β_{ij} be attached to an α_k .

We have just established a way to derive a branched 1-manifold from an order array. To proceed in the opposite direction, we start with an abstract ordered branched 1-manifold K with n (even) branch points and try to build a handle set from this. We must group the k -pronged branch points in pairs for each k :

DEFINITION 3.3. An abstract ordered branched 1-manifold along with information which assigns to each k -pronged branch point x a partner x' , another k -pronged branch point, $\forall k$ is said to be an abstract ordered branched 1-manifold with *pairing information (data)*. We require that $(a')' = a$.

PROPOSITION 3.4. *An abstract ordered branched 1-manifold L with pairing information determines at most one flow equivalence class of associated handle sets; without pairing data, it determines a finite number of such classes.*

PROOF. Given a branched 1-manifold L with n (even) branch points and pairing data, we associate two paired branch points k and k' with the ends of a handle. Then $n/2$ handles have been created, and the number of vertical crossings for each is equal to the number of branches at the corresponding paired branch points; we number these as in Definition 2.16.

To define a potential f —potential in that some inconsistencies may arise that prevent the construction from satisfying certain properties—on this handle set, we shall attempt to draw it in the plane, connecting the crossings with untwisted open strips ($D^1 \times \dot{D}^1$). We use the branched 1-manifold: if there is an edge from branch point k to branch point l , we connect end k to end l , using the cyclic order at the branch points to determine which vertical crossings to connect. This drawing may include undercrossings, but we are only concerned with the abstract handle set, not its immersion in \mathbf{R}^2 . Let B be the matrix for L , and denote vertical crossing numbers by “crossing \sharp .” If

(a)

$$\begin{pmatrix} h \\ l \end{pmatrix} \in b_{ij},$$

connect end i , crossing $\sharp h$, to end j , crossing $\sharp l$;

(b)

$$\begin{pmatrix} h \\ m_j - l + 1 \end{pmatrix} \in b_{ij'},$$

connect end i , crossing $\sharp h$, to end j' , crossing $\sharp l$;

(c)

$$\begin{pmatrix} m_i - h + 1 \\ l \end{pmatrix} \in b_{i'j},$$

connect end i' , crossing $\sharp h$, to end j , crossing $\sharp l$;

(d)

$$\binom{m_i - h + 1}{m_j - l + 1} \in b_{i'j'},$$

connect end i' , crossing $\sharp h$, to end j' , crossing $\sharp l$;

(e)

$$\binom{h}{0} \in b_{ij},$$

draw $f(\text{end } j)$ and connect it to end i , crossing $\sharp h$; likewise for j' replacing j ;

(f)

$$\binom{m_i - h + 1}{0} \in b_{i'j'},$$

draw $f(\text{end } j)$ and connect it to end i' , crossing $\sharp h$; likewise for j' replacing j .

Note that it is immaterial which vertex/end in a pair is labeled “prime,” for interchanging yields the same (potential) handle set, but with a different immersion in \mathbf{R}^2 in our drawing. More precisely, suppose we derive H_1 and f_1 with a certain labeling of L , and H_2 and f_2 arise from the same pairing data with some partners interchanged. Then $f_2 = g \circ f_1$, where g is a diffeomorphism, supported on planar neighborhoods of the handles involved in the interchange, which revolves each of these handles 180° in the plane. g does not affect the order array of the handle set(s). The above is true because the cyclic order of branches at paired branch points determines the 2-dimensional orientation of the associated 1-handle. Interchanging ends just yields different 1-dimensional orientations of leaves of the core and transverse foliations; steps (a)–(f) accommodate this.

For anyone with the patience to read through the above, an example may help. Given L as in Figure 3.4, we form its matrix

$$B = \begin{matrix} & \begin{matrix} 1 & 1' & 2 & 2' \end{matrix} \\ \begin{matrix} 1 \\ 1' \\ 2 \\ 2' \end{matrix} & \begin{pmatrix} \binom{1}{0} & 0 & \binom{2}{0} & 0 \\ 0 & 0 & \binom{1}{1}, \binom{2}{2} & \binom{0}{1} \\ \binom{0}{2} & \binom{1}{1}, \binom{2}{2} & 0 & 0 \\ 0 & \binom{1}{0} & 0 & \binom{2}{0} \end{pmatrix} \end{matrix}.$$

This yields Figure 3.5.

As previously mentioned, the process yields only a potential handle set associated with L , for we may have created some inconsistencies with our added open strips: an annulus, which cannot be the homeomorphic image of a handle; incongruent matches, where a strip connects $f(i)$ to $f(j)$ with $j \neq i'$; or a “reducible” handle set, which does not represent a single basic set. It is the only potential handle set associated with L , for the information contained in L with pairing data forced our construction. The proposition’s second statement follows from the fact that there are only a finite number of ways to pair the branch points of L . Q.E.D.

The pairing data is necessary to ensure a single associated flow equivalence class, as Figure 3.6 shows. Both A and B produce the branched 1-manifold of Figure 3.7.

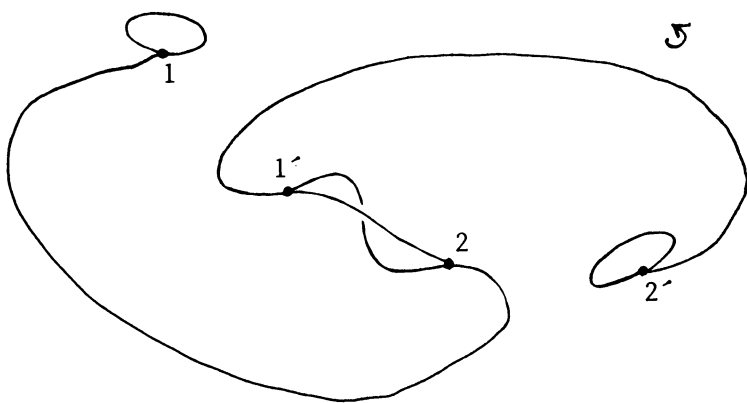


FIGURE 3.4

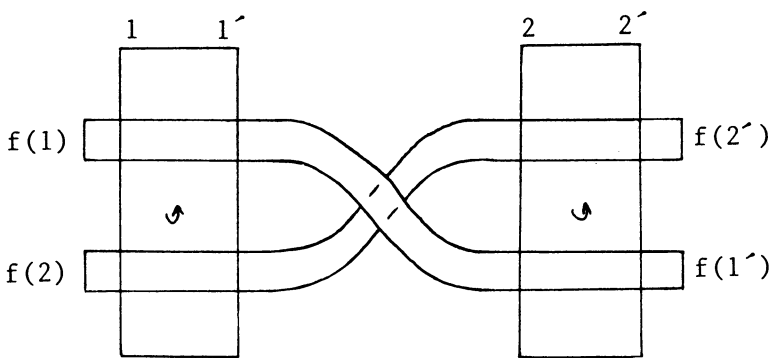


FIGURE 3.5

And yet, using Franks' result again (Theorem 2.10), A is not flow equivalent to B : Writing A as a nonnegative matrix, we get

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

and we find that $\text{Det}(I - A) = -4$. A similar calculation shows that $\text{Det}(I - B) = 4$.

We have just examined how one proceeds from branched 1-manifolds to handle sets, and what amount of additional information is needed to determine these handle sets to varying degrees. One particular piece of additional data which we have not yet analyzed is the “backward” branched 1-manifold: the branched 1-manifold derived from a handle set by considering the reverse flow and its first-return map. One might hope that we could build a unique handle set from a prescribed pair of branched 1-manifolds—one labeled “forward,” the other “reverse”—but Example 3.10 of [3] demonstrates that these two branched 1-manifolds, without pairing data, do not determine even the flow equivalence class of associated handle sets.

Example 3.5:

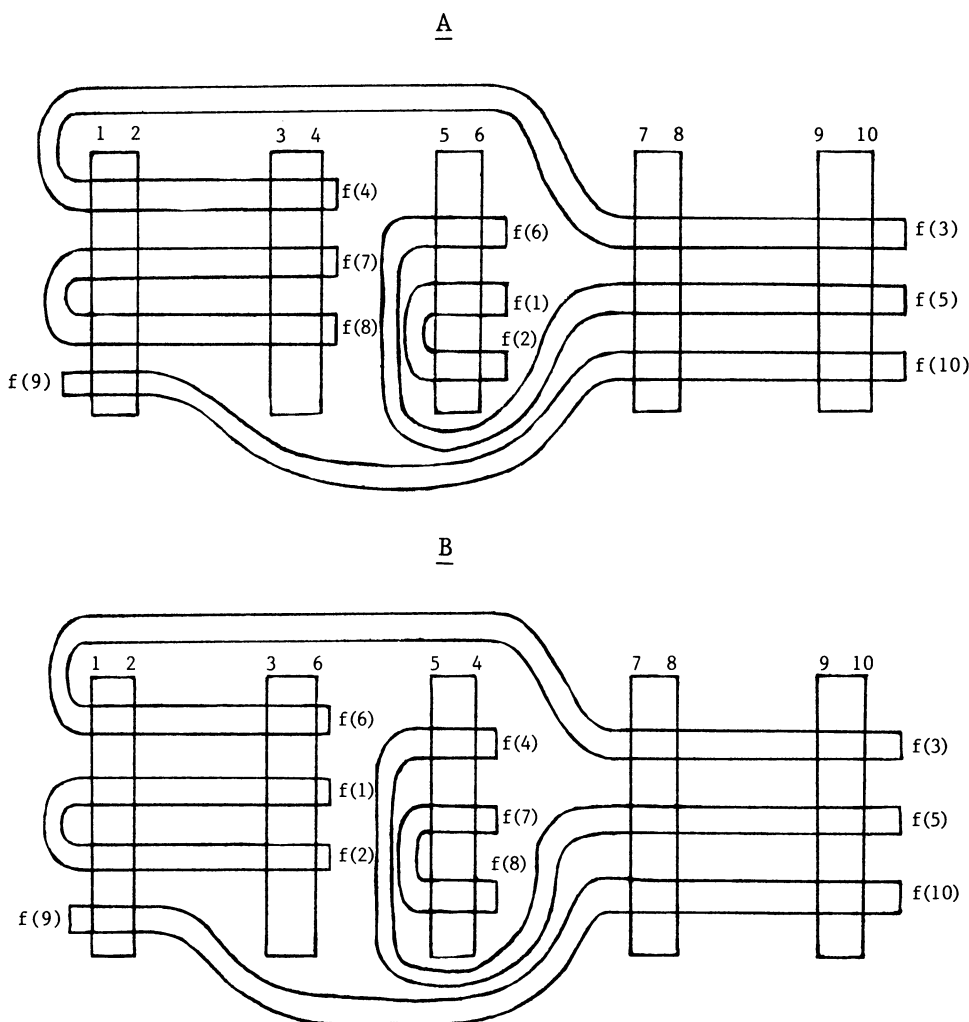


FIGURE 3.6

B. Laminations of handle sets. A major reason for studying branched 1-manifolds in our setting is their strong connection to $W^u(\Lambda)$ ($W^s(\Lambda)$ if we consider the back-based branched 1-manifold), where Λ is the basic set corresponding to our handle set $H = \bigcup_i h_i$.

LEMMA 3.6. *Let Λ' be the suspension of $\Lambda = \bigcup_{n=-\infty}^{\infty} f^n(H)$, where H is a handle set and Λ' is a hyperbolic basic set of ϕ_t , a flow on an oriented 3-manifold M with Lyapunov function g , where $\phi_t|_H = \text{suspension of } f$. Further suppose that $g(\Lambda') = c$ and $S = \text{a component of } g^{-1}(c - \varepsilon)$ for ε sufficiently small (in the same component of $g^{-1}([c - \varepsilon, c + \varepsilon])$ as Λ'). If the order array A describes H , and K is the forward branched 1-manifold derived from A , K carries $W^u(\Lambda') \cap S$. An analogous result holds for $W^s(\Lambda')$.*

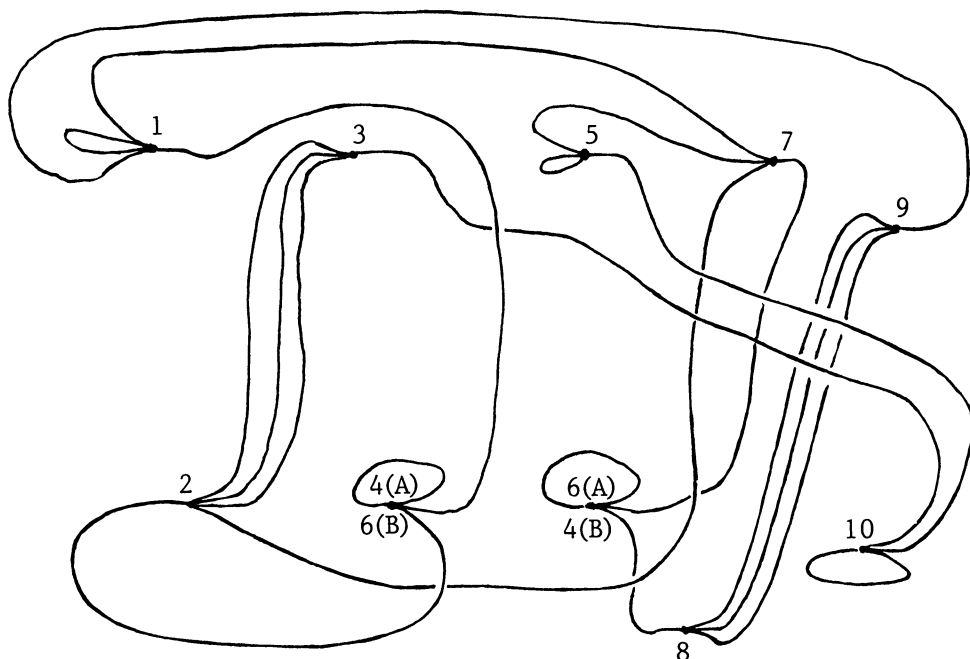


FIGURE 3.7

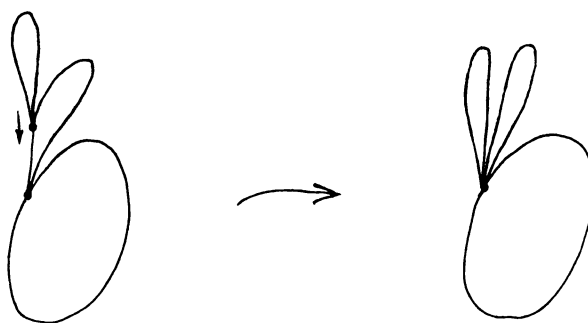


FIGURE 3.8

PROOF. Proposition 2.14 gives us a train track neighborhood $N \subset S$, and we note that $W^u(\Lambda') \cap S \subset N$. For each rectangle k_i in N , we foliate $L_i = \bigcup_{x \in \Lambda} W_i^u(x) \subset k_i$; $L_i \equiv C_i \times I$, where C_i is a Cantor set and $I = [-1, 1]$. Then $W^u(\Lambda') \cap S = \bigcup L_i$. The fact that our branched 1-manifold K carries this lamination follows by noting that K is obtained by collapsing each transverse leaf of N to a point. Q.E.D.

We notice that there is a way to alter a branched 1-manifold K without changing the lamination carried by K : a “combing” process discussed in [8]. It is pictured in Figure 3.8. The associated laminations are identical; we have just discarded portions of the train-track/lamination neighborhood.

We shall make this idea more precise: Start with a branch point x and proceed away from x along its 0-branch to another branch point y (y may equal x), then proceed in a similar manner from y 's 0-branch to branch point z ; continue in this manner until we return to a branch point we have already visited; say, branch point

w . If $w = x$, we do nothing for the moment. If $w \neq x$, we comb: All (nonzero) branches connected to x are attached to y , respecting order, and x (and the path from it to y) are eliminated. We renumber all the branches at y if necessary. This process can be reflected in the branched 1-manifold matrix (see [3]).

Note that the above process does not allow for $\binom{0}{0}$ paths, but we have none in this context.

DEFINITION 3.7. The above manipulation of a branched 1-manifold K is called *combing* K .

We continue combing, dealing with every branch point, until we acquire a finite number of circles along with 1-edged paths connecting them. We may encounter circles containing more than one branch point—there is no canonical way to comb these together. However, we see that the difference between two combing choices is described by a cyclic permutation of the paths connected to the circle, as shown in Figure 3.9. That is, the shift permutation $\sigma(x) = x + 2 \pmod{3}$. Notice that this does not affect either the associated lamination or any paths connected to the inside of the above circle.



FIGURE 3.9

DEFINITION 3.8. *Shift-combing* along a circle in a branched 1-manifold means cyclically permuting all branches on a side (of the two sides) of the circle.

Consider a branched 1-manifold K consisting of circles along with 1-edged paths connecting them, where each circle contains only one branch point. We say that K is in *combed form*. Shift-combing induces an equivalence relation on the set of branched 1-manifolds in combed form: $L_1 \sim L_2$ if L_1 can be shift-combed into L_2 .

The equivalence classes of branched 1-manifolds can help us describe $W^u(\Lambda') \cap S$. Given a representative $L \in [L]$, the train track neighborhood N associated to L is made up of two types of 1-handles: one type is embedded in S , the other has just its interior embedded in S , with one end strictly contained in the other end. Then each nonembedded 1-handle k_i yields a circle in $L_i = \bigcup_{x \in \Lambda} W_i^u(x) \subset k_i$ by a familiar argument involving a contracting map of an interval. Also notice that all other leaves of L_i approach this circle asymptotically as we wind around k_i . The groups of leaves that do this correspond to the embedded 1-handles k_j with at least one end contained in the longer end of k_i . This proves the following.

THEOREM 3.9 (THEOREM B). *Let ϕ_t be a Smale flow on an orientable 3-manifold M with handle set $H = \bigcup h_i$ transverse to ϕ_t and f the first-return map of ϕ_t restricted to H . Suppose $\Lambda' =$ the suspension of $\Lambda = \bigcup_{n=-\infty}^{\infty} f^n(H)$ with corresponding order array A and its derived branched 1-manifold K , and $g(\Lambda') = c$, a critical point of a Lyapunov function g on M . Then for $S =$ a component of $g^{-1}(c - \varepsilon)$, in the same component of $g^{-1}([c - \varepsilon, c + \varepsilon])$ as Λ' , ε sufficiently small, $W^u(\Lambda') \cap S$ is a finite number of circles along with sets of the form $C \times \mathbf{R}$, C a Cantor set, where $C \times t$ approaches a circle asymptotically as $t \rightarrow \infty$ and likewise*

for $t \rightarrow -\infty$; furthermore, $W^u(\Lambda') \cap S$ is carried by K . An analogous result holds for $W^s(\Lambda')$. Q.E.D.

The combing operation on branched 1-manifolds seems natural enough, and this feeling is solidified by the following: given a handle set $H = \bigcup h_i$ and f hyperbolic with respect to H , let A be the corresponding order array and K its derived branched 1-manifold. If we then modify K into K' by squeezing at a pair of branch points (see Figure 3.10) (K' inherits pairing data from A and K) we acquire a new order array A' , with derived branched 1-manifold K' . Let C be the matrix given by c_{ij} = first entry of a_{ij} and C' be the matrix given by c'_{ij} = first entry of a'_{ij} .

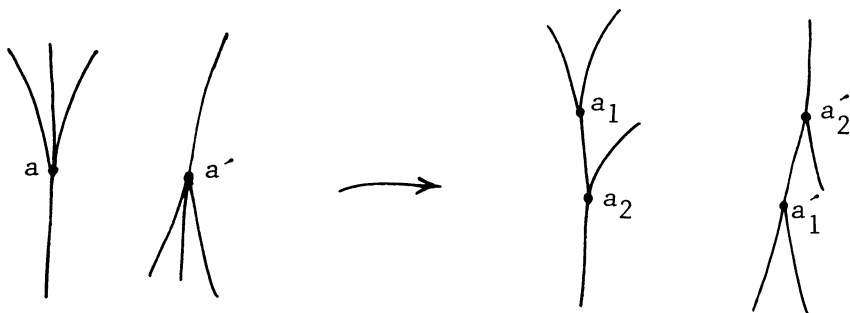


FIGURE 3.10

PROPOSITION 3.10. C and C' are flow equivalent.

PROOF. Assume we have squeezed at branch points a and a' , which correspond to branch rectangle a_0 in the associated thickened template N . Squeezing corresponds to encasing some of the incoming thickened sheets at a_0 in a box-like neighborhood M . However, $X = M - (\text{thickened sheets})$ is wandering, because for each $x \in X$ there is a $T > 0$ such that $\phi_{-t}(x) \notin N$ if $t > T$. Q.E.D.

We now analyze to what extent an equivalence class $[L]$ determines a 1-dimensional basic set. For an arbitrary $[L]$ to uniquely determine a flow equivalence class of associated handle sets, we know that we shall need to add to $[L]$ a type of pairing data, at least. However, this will be slightly different than before: here we label and pair, with "primes," the gaps between consecutive branches at each branch point of some $L \in [L]$ in the expectation, made feasible by Proposition 3.10, that we will squeeze each gap into a 3-pronged branch point. See Figure 3.11. By repeated squeezings, paired gaps become paired (3-pronged) branch points, and the label of each branch point is the label of the (only) gap associated to it. Note that this data for L completely determines such data for any $L' \in [L]$.

DEFINITION 3.11. An equivalence class $[L]$ with the data described above is said to be an *equivalence class with pairing data*.

This is not quite enough; the following shows that we need more information to determine a single flow equivalence class:

$$A = \begin{pmatrix} 0 & 1, 2, 1 & 1, 2, 2 \\ 1, 1, 2 & 1, 3, 1 & 0 \\ 0 & 1, 1, 2 & 1, 1, 1 \end{pmatrix},$$

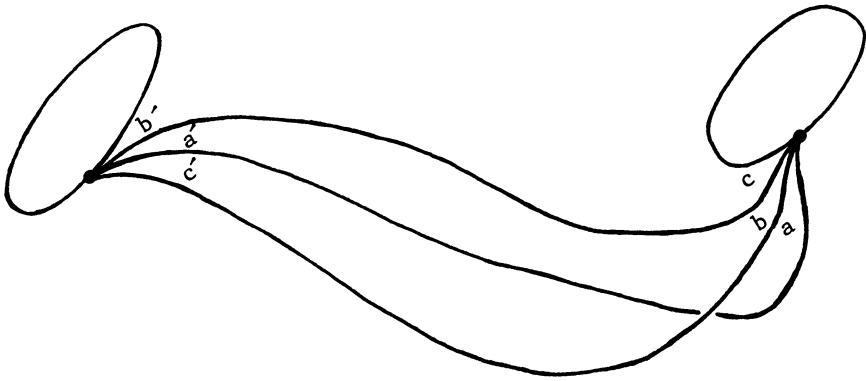


FIGURE 3.11

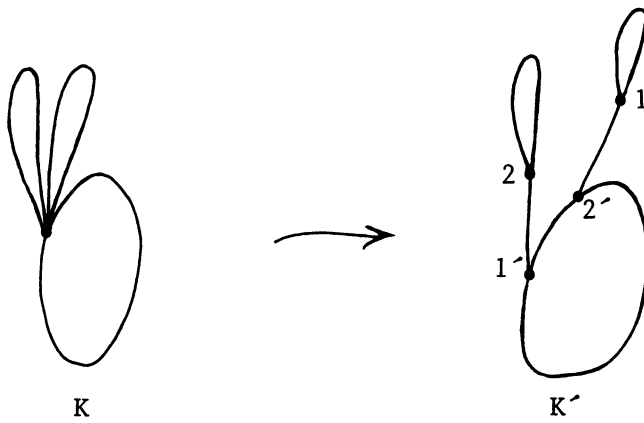


FIGURE 3.12

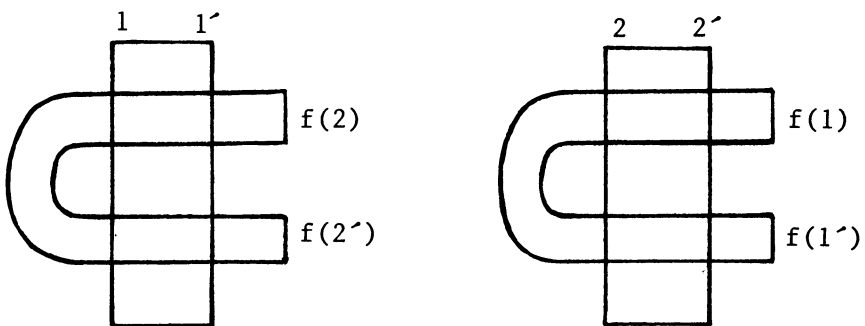


FIGURE 3.13

$$B = \begin{pmatrix} 0 & 1, 1, 1 & 1, 2, 2 \\ 1, 2, 2 & 1, 2, 1 & 0 \\ 1, 1, 2 & 0 & 1, 1, 1 \end{pmatrix}.$$

Both yield Figure 3.11.

Let A' be the matrix given by a'_{ij} = first entry of a_{ij} and B' be the matrix given by b'_{ij} = first entry of b_{ij} . Then $|I - A'| = -1 \neq 0 = |I - B'|$, so by Theorem 2.10, A and B are not flow equivalent.

THEOREM 3.12 (THEOREM C). *An equivalence class $[L]$ with pairing data and squeezing data (numbering each pair of gaps, telling us in which order to squeeze), determines a unique flow equivalence class of the associated handle sets, possibly the empty class.*

PROOF. We start with such an $L \in [L]$. If the number of branch points in L is even and the pairing data on the gaps yields a well-defined pairing data on the branch points—for example, if one branch point has gaps labeled a and b , and another has gaps labeled a' and b' , we can pair these two branch points—we use Proposition 3.4 to find an associated handle set; if we find one, Proposition 3.10 implies that any squeezing of L in pairs will produce a flow-equivalent handle set. If we do not find a handle set from Proposition 3.4 we squeeze L , using the squeezing data to tell us which pair to squeeze, into L' and proceed through the above argument with this. Q.E.D.

It should be noted that by squeezing an unrealizable branched 1-manifold K , we can create a realizable K' . See Figure 3.12. Now K , having an odd number of branch points, cannot be realized as the boundary of a template. However, K' is derived from Figure 3.13.

4. We are now ready to address some embedding issues: ordered branched 1-manifolds in surfaces, and thickened templates in flows on 3-manifolds.

We start with an abstract template corresponding to an order array A . To acquire an associated Smale flow on an oriented 3-manifold, we must, among other things, embed the template's boundary, an ordered branched 1-manifold K , in an oriented surface.

DEFINITION 4.1. The *genus* of K is the least genus of all oriented surfaces in which K can be embedded.

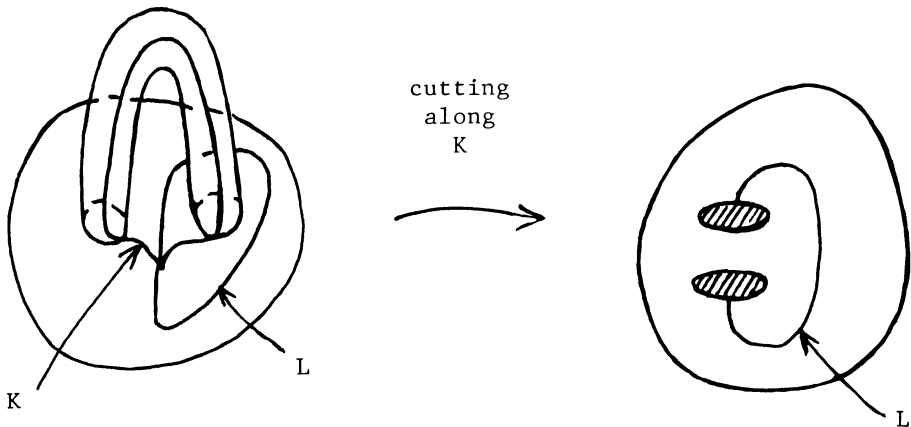


FIGURE 4.1

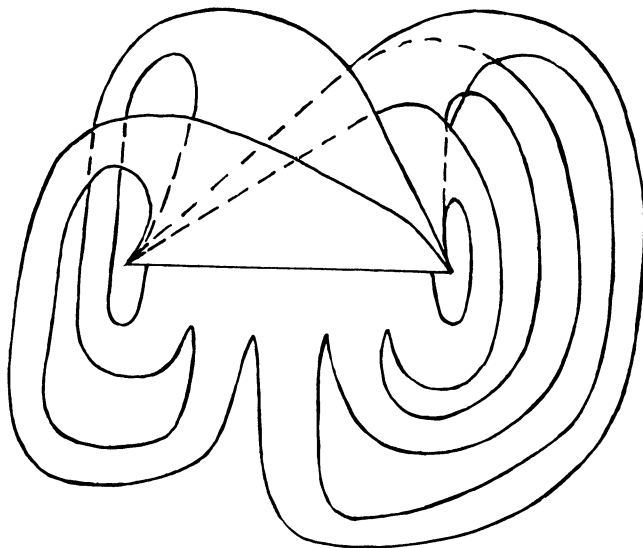


FIGURE 4.2

Note that for any K and K' such that $[\text{combed form of } K] = [\text{combed form of } K']$, genus of $K = \text{genus of } K'$, for combing can be done with an arbitrarily small C^1 -perturbation.

THEOREM 4.2. *Suppose that ϕ_t is a flow on an orientable 3-manifold M with Lyapunov function g and Λ' is a 1-dimensional hyperbolic basic set of ϕ_t , so that Λ' is the suspension of $\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(H)$, where H is a handle set and $\phi_t|_H = \text{suspension of } f$. Let $g(\Lambda') = c$ and pick $\varepsilon > 0$ sufficiently small so that there is no other basic set in the component of $g^{-1}([c - \varepsilon, c + \varepsilon])$ which contains Λ' ; call this component X , and let S_1, \dots, S_m be the components of $g^{-1}(c - \varepsilon)$ in X . If A is the order array associated to Λ and K is its derived branched 1-manifold, which decomposes into components K_1, \dots, K_j , then $g(S_i) \geq \sum_t g(K_t)$ with t ranging through all values such that K_t is embedded in S_i , where $g(\cdot)$ denotes "genus of." Furthermore, $m \leq j$.*

PROOF. m cannot exceed j , for each component S_i must intersect $W^u(\Lambda')$, and each K_h , being connected, must lie in a single S_i .

As for the first statement, pick an arbitrary S_i . Up to homeomorphism, we can consider S_i to be S^2 with $g(S_i)$ handles attached in an untwisted, unlinked way. Suppose K_{t_1}, \dots, K_{t_l} are the components of K in S_i . By cutting along K_{t_1} as in Figure 4.1 at the $g(K_{t_1})$ handles of S_i which intersect K_{t_1} , we turn S_i into S_i^1 , a surface with boundary where $g(S_i^1) \leq g(S_i) - g(K_{t_1})$.

Because the K_j are disjoint, K_{t_2}, \dots, K_{t_l} are embedded in S_i^1 . Repeating this process for each K_{t_j} , $2 \leq j \leq l$, yields S_i^l where $0 \leq g(S_i^l) \leq g(S_i) - \sum_{j=1}^l g(K_{t_j})$. The desired inequality follows. Q.E.D.

We say that an order array A can be realized in a flow ϕ_t if A corresponds to a basic set of ϕ_t as in the hypotheses of the preceding theorem. When a branched 1-manifold K decomposes into K_1, \dots, K_j , we call $\max_{1 \leq i \leq j} \{\text{genus of } K_i\}$ the *genus of K* .

THEOREM 4.3 (THEOREM A). *Let A be an order array with derived branched 1-manifold K , and suppose that ϕ_t is a nonsingular Smale flow on a compact oriented 3-manifold M , with $H_1(M; \mathbb{Q}) = 0$. A necessary condition for A to be realized in ϕ_t is that the genus of $K \leq 1$.*

PROOF. Lemma 2.2 of [6] implies that the surfaces S of Theorem 4.2 are each tori: the Poincaré-Hopf formula ensures this, because the S_i are transverse to a nonsingular flow. Our result then follows from Theorem 4.2. Q.E.D.

Theorem 4.3 is a simple extension of Theorem 2 in [4] to the set of order arrays. Consider the template T in Figure 4.2. Let $K = \partial T$. T corresponds to the structure matrix $A = [5]$, so $Z/(I - A)Z \cong Z_4$ is cyclic; therefore there is no obstruction to its embedding in a nonsingular Smale flow on S^3 from the earlier theorem.

Modifying as in Figure 2.3, we describe T by the order array

$$A' = \begin{pmatrix} 1, 1, 4 & 1, 1, 5 & 1, 1, 2 & 1, 1, 1 & 1, 1, 3 \\ 1, 2, 4 & 1, 2, 5 & 1, 2, 2 & 1, 2, 1 & 1, 2, 3 \\ 1, 3, 4 & 1, 3, 5 & 1, 3, 2 & 1, 3, 1 & 1, 3, 3 \\ 1, 4, 4 & 1, 4, 5 & 1, 4, 2 & 1, 4, 1 & 1, 4, 3 \\ 1, 5, 4 & 1, 5, 5 & 1, 5, 2 & 1, 5, 1 & 1, 5, 3 \end{pmatrix},$$

which has derived branched 1-manifold K' ; note that genus of $K' =$ genus of K . The matrix of K is

$$\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right),$$

which has genus 2. Therefore A' cannot be realized in a nonsingular Smale flow on S^3 .

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